THE FEDOSOV MANIFOLDS AND MAGNETIC MONOPOLE

Alexander S. Ushakov

St.Petersburg State University, V.A.Fock Institute of Physics. Ulianovskaya ul. 1, Petrodvoretz, Saint-Petersburg, Russia. asushakov@gmail.com

Abstract

A non-symplectic generalization of Hamiltonian mechanics is considered. It allows include into consideration "non-Lagrange" systems, such as theory of charged particle in the field of magnetic monopole. The corresponding generalization for the Fedosov manifolds is given. The structure of phase space of "charged particle in the field of magnetic monopole" is studied.

1 Introduction

There are equations of motion in mechanics, that cannot be obtained in the framework of Lagrangian formalism, for example [1, p.13]:

$$\ddot{q}_i = \alpha e_{ijk} q_j \dot{q}_k, \quad i,j,k = 1,2,3, \quad \alpha = const, \tag{1}$$

where e_{ijk} – is absolutely antisymmetric tensor, $e_{123} = 1$. The Hamiltonian formalism allows to deal with more general class of dynamical systems. All the information about system is contained in two objects – the Hamilton function and symplectic 2-form, while in Lagrangian mechanics only one function (Lagrangian) is necessary. The action functional in the Lagrange theory is varied on n functions of time $q^i(t)$, but in Hamiltonian formalism — on $2n$ functions $(q^i(t), p_i(t))$. Usually the simplest case is used in Hamiltonian mechanics, when symplectic form can be transformed to standard one by the Darboux transformation. These dynamical systems allows nontrivial generalization by choosing different 2-form [2]. Equations (1) can obtained from the Hamiltonian equations of generalized mechanics under the appropriate choice of 2-form .

To define a dynamical system, in Hamiltonian mechanics one needs:

1) Even-dimensional manifold M^{2n} and the closed non-degenerated 2-form on it $\omega^2 = \sum_{\mu>\nu} \omega_{\mu\nu}(x)\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$, $\mu,\nu = 1,2,...,2n$, $x \in M^{2n}$, $\mathrm{d}\omega^2 = 0$. The non-degeneracy of 2-form means that $\det(\omega_{\mu\nu}(x)) \neq 0$; and in this case the manifold M^{2n} is orientable. The manifold M^{2n} equipped with ω^2 is called symplectic [3, 4].

Differential 2-form allows us to define Poissons bracket for a pair of functions $f(x)$, $g(x)$:

$$\{f, g\} \equiv \frac{\partial f}{\partial x^\mu} \omega^{\mu\nu} \frac{\partial g}{\partial x^\nu} \equiv \omega^{\mu\nu} \partial_\mu f \partial_\nu g \,,$$

$$\omega^{\mu\nu} \omega_{\nu\alpha} = \delta^\mu_\alpha.$$

2) Some function (Hamiltonian) of the local variables $H(x) \equiv H(p,\, q)$ on M^{2n}, where $x = (q^1, ..., q^n, p_1, ..., p_n)$, q^i and p_i – are generalized coordinates and momenta correspondingly.

3) One has to postulate the existence of "time" (positive parameter t); $q = q(t)$, $p = p(t)$ are single–valued functions of t.

4) One has to define equations of motion

$$\dot{x}^\mu = \{x^\mu, H\} = \omega^{\mu\nu} \partial_\nu H \,, \quad \dot{x}^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}t} \,.$$

Besides obvious properties of Poissons brackets (linearity with respect to each argument and antisymmetry) existence of the Jacoby identity is supposed

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0, \tag{2}$$

that is equivalent to the fact that 2-form is closed

$$\partial_\alpha \omega_{\mu\nu} + \partial_\mu \omega_{\nu\alpha} + \partial_\nu \omega_{\alpha\mu} = 0. \tag{3}$$

Notice, that the orientation of phase space defines "the arrow of time", namely, the Hamiltonian equations of motion are not invariant to reflection of time in contrast to the standard Lagrangian ones. Only simultaneous change of the matrix $\omega_{\mu\nu}$ (change the orientation) and the reversal of time does not change the equation of motion.

It is easy to obtain equation (1) by the simplest generalization of Hamilton mechanics, namely by the proper choice of 2-form. For a free particle ($H = \vec{p}^2/2$) one can consider the antisymmetric 6×6 matrix

$$\omega^{\mu\nu} = \begin{pmatrix} g e_{ijk} p_k & \delta_{is} \\ -\delta_{rj} & f e_{rsk} q_k \end{pmatrix} \,, \tag{4}$$

where $f = f(q)$, $g = g(p)$ are arbitrary functions; $i, j, r, s, k = 1, 2, 3$. Then the Hamiltonian equations of motion

$$\dot{q}_i = p_i \,, \quad \dot{p}_i = f e_{ijk} q_j p_k \tag{5}$$

are equivalent to the equations (1), if $f = \alpha$. Equations (5) with the specific choice of f describe motion of charged particle in the field of magnetic monopole (see Sec. 2).

Deformation quantization [5] is formulated in terms of symplectic manifolds supplied with symmetric connection consistent with the given symplectic structure. In the paper [6], devoted to study of the properties of symplectic curvature tensor, the term "Fedosov manifolds" is proposed. We will use it both in the case of non-closed 2-forms (non-symplectic manifolds) and (as a result) in the case of non-symmetric connections (see Sec. 3).

In this paper we analyse generalization of the non-symplectic Hamiltonian mechanics. Structure of phase spaces is studied with the help of generalized Fedosov construction. Curvature tensor and torsion tensor of given phase space of "charged particle in the field of magnetic monopole" is not zero.

In section 2 main peculiarities of non-symplectic Hamilton mechanics are formulated. It allows to obtain "non-Lagrange" equations of motion, particularly the motion of electric charged in the field of magnetic monopole. In section 3 generalization of the Fedosov manifolds for non-symplectic Hamilton mechanics is constructed. Section 4 is dedicated to study of space phase structure, 2-form of which is set by matrix (6). Corresponding 2-form allows to take into consideration magnetic monopole. Final remarks are given in the section 5.

2 Peculiarities of generalized Hamiltonian mechanics and magnetic monopole

One of peculiarities of given generalization of Hamiltonian mechanics is that 2-form defined by the matrix inverse to (4)

$$\omega_{\mu\nu} = \frac{1}{(1 - fgq_k p_k)} \begin{pmatrix} fe_{ijk}q_k & fgp_i q_s - \delta_{is} \\ \delta_{rj} - fgq_r p_j & ge_{rsk}p_k \end{pmatrix}, \tag{6}$$

is not closed: $\mathrm{d}\omega^2 \neq 0$. So, one obtains non-symplectic Hamiltonian mechanics. But what this peculiarity leads to? Firstly, the Jacoby identity (2) is not satisfied, i. e. the Lie algebra structure of the Poisson brackets on the surface of smooth functions is broken. Secondly, the Darboux theorem is inapplicable, since for its proof the Jacoby identity is needed [4]. Thirdly, the

variational principle for these theories is not known (for non-Lagrange systems see [7]; for inexact forms see paper [8], but closed form is needed there (if one considers some star domain then any closed form on it is exact); also see [9]). Fourthly, the Hamiltonian phase flow does not preserve the Poincare invariants besides phase volume (the Liouville theorem). To make sure of it let us consider the element of phase space $\Omega_k = (\omega^2)^k$, $k = 1, 2, ..., n$; its differential

$$\mathrm{d}\Omega_k = k(\omega^2)^{k-1}\mathrm{d}\omega^2$$

is zero only in two cases: either form ω^2 is closed or $k = n$ ($\mathrm{d}\Omega_n$ is $(2n+1)$-form, forms of this type are absent in space M^{2n}, only if $\partial\omega_{\mu\nu}/\partial t = 0$; otherwise the Liouville theorem is failed). Obviously, if $\mathrm{d}\omega^2 = 0$, then $d\Omega_k/dt = 0$ irrespective of the choice of Hamiltonian, what proves the theorem about the Poincare invariants [3].

Even if $g = 0$ in (6), one obtaines the same equations (5) (Hamiltonian is the same), although 2-form has changed:

$$\omega^2 = \mathrm{d}p_i \wedge \mathrm{d}q^i + f e_{ijk} q^k \, \mathrm{d}q^i \wedge \mathrm{d}q^j. \tag{7}$$

Equations (5) allow interesting interpretation. If $f = eg_e/(mc(q_1^2 + q_2^2 + q_3^2)^{\frac{3}{2}})$, $q_i = r_i$, then one obtains [10][1]

$$\ddot{\vec{r}} = \frac{eg_e}{mc} \frac{\vec{r} \times \dot{\vec{r}}}{|r|^3}. \tag{8}$$

Eq. (8) describes motion of the particle with charge e and mass m in the field of magnetic monopole with magnetic charge g_e (c – is the velocity of light). But at the same time (8) are the Hamiltonian equations of motion for free particle ($H = \vec{p}^2/2$) with modified symplectic form (7). Actually in the case of $f = \lambda(q_1^2 + q_2^2 + q_3^2)^{\frac{3}{2}}$ (λ is some constant) 2-form (7) is closed everywhere, except of one point: $d\omega^2 = 4\pi eg_e\delta(\vec{r})/(mc)$, where $\delta(\vec{r})$ is Dirac delta function. Notice, that in paper [12] magnetic monopoles are considered from the point of view of string theory, and the variational principle is suggested.

Equations (8) could be obtained not only by modification of symplectic structure. According to Cabibbo and Ferrari [11] to include magnetic monopoles in the Maxwell theory one should present electromagnetic force tensor $F_{\mu\nu}$ in the form

$$\begin{gathered} F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + e_{\mu\nu\rho\sigma}\partial_\rho B_\sigma, \\ \mu, \nu, \rho, \sigma = 1, 2, 3, 4, \end{gathered} \tag{9}$$

[1] Poincare instead of eg_e/mc used λ.

where $e_{\mu\nu\rho\sigma}$ – is unit skew-symmetric tensor, $e_{1234} = 1$, and B_σ – is the second vector field. Expression (9) comes to ordinary $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ if the last term vanishes ($B_\sigma = const$).

The second vector potential B_μ increases the gauge transformation group. The variational principle is not found this case.

In paper [13] equations (5) are obtained in terms of Hamiltonian mechanics with constrains and the standard symplectic matrix

$$\omega^{\mu\nu} = \begin{pmatrix} 0 & \delta_{rs} \\ -\delta_{rs} & 0 \end{pmatrix}, \quad r, s = 1, 2, 3, 4, 5, 6;$$

dimension of phase space here is $2n = 12$. Consider the Hamiltonian

$$H = f e_{ijk} p_j q_k q_{i+3} + \frac{1}{2}\delta_{ij}\left(p_i p_j - p_{i+3} p_{j+3}\right), \quad i, j, k = 1, 2, 3,$$

where $f = f(q_1, q_2, q_3)$. Equations of motion are:

$$\dot{q}_i = f e_{ijk} q_j q_{k+3} + p_i, \quad \dot{p}_i = f e_{ijk} p_j q_{k+3} + \partial_{q_i} f\, e_{ijk} q_i p_j q_{k+3},$$

$$\dot{q}_{i+3} = -p_{i+3}, \quad \dot{p}_{i+3} = f e_{ijk} p_k q_j, \quad i, j, k = 1, 2, 3.$$

Let q_i and p_i obey the constraints

$$\phi_i = p_i + p_{i+3} = 0, \quad \phi_{i+3} = q_i - q_{i+3} = 0, \quad i = 1, 2, 3.$$

They are constrains of the 1st kind:

$$\{\phi_i, H\} = f e_{ijk} q_j p_k + f e_{ijk} p_j q_{k+3} + \partial_{q_i} f\, e_{ijk} q_i p_j q_{k+3} \approx 0,$$

$$\{\phi_{i+3}, H\} = f e_{ijk} q_j q_{k+3} + p_i + p_{i+3} \approx 0, \quad \{\phi_r, \phi_s\} = 0,$$
$$i, j, k = 1, 2, 3, \quad r, s = 1, 2, ..., 6.$$

It is easy to see that equations of motion of physical variables are indentical to Eq. (5). The Hamiltonian $H \approx 0$ – rather unusual property for mechanics with constraints [1, p. 140].

There exists a system similar to the previous one but with the Hamiltonian

$$\widetilde{H} = f e_{ijk} p_j q_k q_{i+3} + \frac{1}{2}\delta_{ij}\left(p_i p_j + p_{i+3} p_{j+3}\right), \quad i, j, k = 1, 2, 3,$$

which is not equal to zero in the weak sence. Equations of motion are the same except the equations $\dot{q}_{i+3} = p_{i+3}$. The constrains read

$$\widetilde{\phi}_i = p_i + p_{i+3} = 0, \quad \widetilde{\phi}_{i+3} = q_i^2 - q_{i+3}^2 = 0, \quad i = 1, 2, 3.$$

They are constrains of 1st kind (no sum over i):

$$\{\widetilde{\phi}_i, \widetilde{H}\} = f e_{ijk} q_j p_k + f e_{ijk} p_j q_{k+3} + \partial_{q_i} f \, e_{ijk} q_i p_j q_{k+3} \approx 0,$$
$$\{\widetilde{\phi}_{i+3}, \widetilde{H}\} = 2 f q_i e_{ijk} q_j q_{k+3} + 2 q_i p_i - 2 q_{i+3} p_{i+3} \approx 0, \quad \{\widetilde{\phi}_r, \widetilde{\phi}_s\} \approx 0,$$
$$i, j, k = 1, 2, 3, \quad r, s = 1, 2, ..., 6.$$

Equations of motion for physical variables coincide with Eqs. (5). Taking into consideration constraints we see that the Hamiltonian $\widetilde{H}$ is not equal to zero. Notice that unlike theories considered usually each constraint $\widetilde{\phi}_{i+3} \approx 0$ have two different solutions.

3 Fedosov's manifolds

Let us consider a manifold M^{2n} with non-degenerate 2-form ω^2. Let us determine covariant derivative ∇ on M^{2n}, consistent with 2-form ω^2, i.e. $\nabla\omega^2 = 0$ [5]. In the coordinate basis this condition means that

$$\nabla_k \omega_{\mu\nu} = \partial_k \omega_{\mu\nu} - \Gamma^l_{\mu k} \omega_{l\nu} - \Gamma^l_{\nu k} \omega_{\mu l} = 0. \tag{10}$$

Let's introduce $\omega_{\nu l} \Gamma^l_{\mu k} = \Gamma_{\nu,\mu k}$ and write Eq. (10) together with those, obtained by the cyclic permutation of indices

$$\partial_k \omega_{\mu\nu} = -\Gamma_{\nu,\mu k} + \Gamma_{\mu,\nu k}, \tag{11}$$

$$\partial_\mu \omega_{\nu k} = -\Gamma_{k,\nu\mu} + \Gamma_{\nu,k\mu}. \tag{12}$$

$$\partial_\nu \omega_{k\mu} = -\Gamma_{\mu,k\nu} + \Gamma_{k,\mu\nu}, \tag{13}$$

Adding expressions (12) (13), and subtracting (11), one obtains the following representation for $\Gamma_{k,\mu\nu}$:

$$\Gamma_{k,\mu\nu} = \frac{1}{2}\big(\partial_\nu \omega_{k\mu} + \partial_\mu \omega_{\nu k} - \partial_k \omega_{\mu\nu}\big) + S_{\mu\nu k} + S_{k\mu\nu} - S_{\nu k\mu}, \tag{14}$$

where $S_{k\mu\nu}$ is symmetric part of connection $\Gamma_{k,\mu\nu}$

$$S_{k\mu\nu} = \frac{1}{2}\big(\Gamma_{k,\mu\nu} + \Gamma_{k,\nu\mu}\big). \tag{15}$$

If 2-form ω^2 is not closed (e.g., if it is given by matrix (6)), then condition (10) imposes a restriction on symmetry of indices of symplectic connection $\Gamma_{k,\mu\nu}$, namely, connections cannot be symmetric. To make sure of this fact let's us sum up equations (11)–(13) and take into consideration the fact that the identity (3) is not satisfied

$$\partial_k\omega_{\mu\nu} + \partial_\nu\omega_{k\mu} + \partial_\mu\omega_{\nu k} = 2T_{k\mu\nu} + 2T_{\mu\nu k} + 2T_{\nu k\mu} \neq 0, \tag{16}$$

where $T_{k\mu\nu} = \frac{1}{2}\big(\Gamma_{k,\mu\nu} - \Gamma_{k,\nu\mu}\big)$ – is the antisymmetric part of connection $\Gamma_{k,\mu\nu}$. From (16) it follows that connections could not be symmetric.

As it has been pointed in introduction we will use term the Fedosov manifolds also to manifolds with given non-closed 2-form.

Curvature tensor R^i_{qkl} is determined by action of the commutator of covariant derivatives $[\nabla_k, \nabla_l] = \nabla_k\nabla_l - \nabla_l\nabla_k$ on a vector field a^i. The first term reads:

$$\nabla_k\nabla_l a^i = \nabla_k\big(\partial_l a^i + \Gamma^i_{ql}a^q\big) =$$
$$= \partial_k\big(\partial_l a^i + \Gamma^i_{ql}a^q\big) + \Gamma^i_{pk}\big(\partial_l a^p + \Gamma^p_{ql}a^q\big) - \Gamma^p_{lk}\big(\partial_p a^i + \Gamma^i_{qp}a^q\big) =$$
$$\partial_k\partial_l a^i + \Gamma^i_{ql}\partial_k a^q + \Gamma^i_{pk}\partial_l a^p - \Gamma^p_{lk}\partial_p a^i + a^q\partial_k\Gamma^i_{ql} + \Gamma^i_{pk}\Gamma^p_{ql}a^q - \Gamma^p_{lk}\Gamma^i_{qp}a^q.$$

Then the commutator gives

$$[\nabla_k, \nabla_l]a^i = -R^i_{qkl}a^q + T^p_{kl}\partial_p a^i, \tag{17}$$

where

$$R^i_{qkl} = \partial_l\Gamma^i_{qk} - \partial_k\Gamma^i_{ql} + \Gamma^i_{pl}\Gamma^p_{qk} - \Gamma^i_{pk}\Gamma^p_{ql}, \tag{18}$$

is the curvature tensor;

$$T^p_{kl} = \Gamma^p_{kl} - \Gamma^p_{lk} \tag{19}$$

is the torsion tensor.

Curvature tensor is antisymmetric with respect to lower indices $R^i_{qkl} = -R^i_{qlk}$.

Let us introduce tensor $R_{jqkl} = \omega_{ji} R^i_{qkl}$, which is also antisymmetric with respect to lower indices

$$R_{jqkl} = -R_{jqlk}. \tag{20}$$

From the definition of $\Gamma_{j,qk}$ and expression (11) it follows that

$$\omega_{ji} \partial_l \Gamma^i_{qk} = \partial_l \Gamma_{j,qk} - \partial_l \omega_{ji} \Gamma^i_{qk} = \partial_l \Gamma_{j,qk} + \big(\Gamma_{i,jl} - \Gamma_{j,il}\big) \Gamma^i_{qk}.$$

So, taking into consideration (18) one gets

$$R_{jqkl} = \partial_l \Gamma_{j,qk} - \partial_k \Gamma_{j,ql} + \Gamma_{i,jl} \Gamma^i_{qk} - \Gamma_{i,jk} \Gamma^i_{ql}. \tag{21}$$

Using (11) one obtains following relations

$$\partial_l \partial_k \omega_{jq} = \partial_l \Gamma_{j,qk} - \partial_l \Gamma_{q,jk},$$
$$\partial_k \partial_l \omega_{jq} = \partial_k \Gamma_{j,ql} - \partial_k \Gamma_{q,jl}.$$

The latter equations lead to

$$\partial_l \Gamma_{j,qk} - \partial_k \Gamma_{j,ql} = \partial_l \Gamma_{q,jk} - \partial_k \Gamma_{q,jl}. \tag{22}$$

Using relation (22) and equality

$$\Gamma_{i,jl} \Gamma^i_{qk} = -\omega_{si} \Gamma^s_{jl} \Gamma^i_{qk} = -\Gamma^s_{jl} \Gamma_{s,qk},$$

one obtains symmetry property of tensor R_{jqkl} in respect to the first two indices:

$$R_{jqkl} = R_{qjkl}. \tag{23}$$

From (23) it follows that

$$R^i_{ikl} = \omega^{ip} R_{pikl} = 0.$$

Let us now introduce the Ricci tensor for Fedosov's manifold:

$$R_{ql} = R^i_{qil} = \omega^{ik} R_{kqil}. \tag{24}$$

Notice, that in the case of symmetric connections Ricci tensor is also symmetric [6].

Let us introduce scalar curvature for Fedosov manifold:

$$R = \omega^{ik} R_{ki}. \tag{25}$$

Theorem 1. For any Fedosov's manifold scalar curvature $R = 0$.
Proof. Using the definitions (24) and (25)

$$R = \omega^{lk} R_{kl} = \omega^{lk}\omega^{ij} R_{jkil} = \frac{1}{2}\big(\omega^{lk}\omega^{ij} R_{jkil} + \omega^{lk}\omega^{ji} R_{ikjl}\big),$$

and taking into account symmetry properties of the curvature tensor (20) (23), one obtains

$$R = \frac{1}{2}\big(\omega^{lk}\omega^{ij} R_{jkil} - \omega^{lk}\omega^{ji} R_{kilj}\big) \equiv 0.$$

The theorem is proved.

4 Magnetic monopole

Now we can begin study of the phase space structure of such systems as charged particle in the field of magnetic monopole. Firstly let's consider 2-form given by matrix (6). From Sec.3 we know that space has torsion and zero scalar curvature. The curvature tensor is not uniquely defined by the connection consistent with the 2-form. The symmetric part of the connection doesn't depend on the choice of the 2-form, hence one can consider only the part of the curvature tensor associated with the skew-symmetric part of the connection.

Theorem 2. If 2-form ω^2 is non-degenerate, then there exists unique connection that is skew-symmetric and consistent with this 2-form ω^2. This connection in coordinate bases is given by:

$$\Gamma^l_{\mu\nu} = \frac{1}{2}\omega^{lk}\big(\partial_\nu \omega_{k\mu} + \partial_\mu \omega_{\nu k} - \partial_k \omega_{\mu\nu}\big). \tag{26}$$

Proof. Rising index k in expression (14) (that is consequence of consistency of connection with 2-form (10)) and taking into consideration that $\Gamma_{k,\mu\nu} = -\Gamma_{k,\nu\mu}$, one obtains required formula. Theorem is proved.

Notice that the symmetric part of the connection doesn't transform as tensor under the coordinate transformations, hence in new coordinates the connection, in general, can have the symmetric part.

Consider the specific coordinate bases in which the connection is skew-symmetric. Then one can calculate the curvature in this bases by (21). It is the tensor with respect to the coordinate transformations, hence the curvature in some other coordinate bases can be obtained from the curvature in

the chosen specific bases by the tensor law, irrespective of possible break of the skew-symmetricity of the connection.

If we consider the model with skew-symmetric connections only, some additional constraints should be imposed. For example, we can chose some specific coordinate system and allow only linear transformations, or equip the manifold with the Riemann (Pseudo-Riemann) metrics in addition to the 2-form. This metric can be coordinated with the 2-form by the almost complex structure which always exists on the even-dimensional orientable manifold. Although metrics and almost complex structure is connected with the 2-form in non-invariant way, it is convenient to use them together [3]. Then we require the connection to be coordinated both with the 2-form and metrics. Then the metrics will be connected with the symmetric part of the connection, while the 2-form - with the skew-symmetric part.

Let's show that in this case curvature tensor is not zero. It is sufficient to prove this fact for any element of R_{ijkl}. We will consider R_{1112}. Taking into account (26) and (21) one has

$$\begin{aligned} R_{1112} &= -\partial_{q_1}\Gamma_{112} = -\partial_{q_1}\partial_{q_1}\omega_{21} = -\partial_{q_1}\left(\frac{q_3(\partial_{q_1}f + f^2 g p_1)}{(1 - fgq^i p_i)^2}\right) = \\ &= q_3 \cdot \frac{\partial_{q_1}\partial_{q_1}f \cdot (1 - fgq^i p_i) + 2(\partial_{q_1}f)^2 g(q^i p_i) + 3\partial_{q_1}f \cdot fgp_1 + f^3 g^2 p_1^2}{(1 - fgq^i p_i)^3}, \end{aligned} \tag{27}$$

where in the case of magnetic monopole

$$f = (q_1^2 + q_2^2 + q_3^2)^{-\frac{3}{2}}, \quad g = (p_1^2 + p_2^2 + p_3^2)^{-\frac{3}{2}}, \tag{28}$$

$$\partial_{q_1}f = \frac{-3q_1}{(q_1^2 + q_2^2 + q_3^2)^{\frac{5}{2}}}, \quad \partial_{q_1}\partial_{q_1}f = \frac{3(4q_1^2 - q_2^2 - q_3^2)}{(q_1^2 + q_2^2 + q_3^2)^{\frac{7}{2}}}.$$

For the Ricci tensor in case of 2-form (6), taking into consideration (4) and (24) one obtains

$$\begin{aligned} R_{kl} = \omega^{ij}R_{jkil} &= gp_3(R_{2k1l} - R_{1k2l}) + gp_2(R_{1k3l} - R_{3k1l}) + gp_1(R_{3k2l} - R_{2k3l}) + \\ &+ (R_{4k1l} - R_{1k4l}) + (R_{5k2l} - R_{2k5l}) + (R_{6k3l} - R_{3k6l}) + \\ &+ fq_3(R_{5k4l} - R_{4k5l}) + fq_2(R_{4k6l} - R_{6k4l}) + fq_1(R_{6k5l} - R_{5k6l}). \end{aligned} \tag{29}$$

We are not going to determine all elements of Ricci tensor (29), and limit ourselves only by diagonal elements R_{ll}. Using (21) and definition $\Gamma_{r,kl} = \omega_{ri}\Gamma^i_{kl}$ we have:

$$R_{ll} = \omega^{kj}\partial_l\Gamma_{j,lk} + \omega^{kj}\omega^{ir}\Gamma_{i,jl}\Gamma_{r,lk}. \tag{30}$$

According to (26) the first term here is

$$\omega^{kj}\partial_l\Gamma_{j,lk} = \frac{1}{2}\omega^{kj}\partial_l\left(\partial_k\omega_{jl} + \partial_l\omega_{kj} - \partial_j\omega_{lk}\right) =$$
$$\frac{1}{2}\omega^{kj}\partial_l\partial_l\omega_{kj} + \frac{1}{2}\left(\omega^{kj}\partial_l\partial_k\omega_{jl} - \omega^{jk}\partial_l\partial_j\omega_{kl}\right) = \frac{1}{2}\omega^{kj}\partial_l\partial_l\omega_{kj}. \quad (31)$$

From $\omega^{ri}\omega_{ij} = \delta^r_j$ it follows

$$\omega^{ri}\partial_l\omega_{ij} = -\omega_{ij}\partial_l\omega^{ri}. \quad (32)$$

To find the second term (30) one uses Eqs. (26) and (32):

$$\omega^{kj}\omega^{ir}\Gamma_{i,jl}\Gamma_{r,lk} = \frac{1}{4}\omega^{kj}\left(-\omega_{ij}\partial_l\omega^{ri} + \omega^{ri}\partial_j\omega_{li} + \omega_{jl}\partial_i\omega^{ri}\right)\Gamma_{r,kl} =$$
$$= \frac{1}{4}\left(\delta^k_i\partial_l\omega^{ri} + \omega^{kj}\omega^{ri}\partial_j\omega_{li} + \delta^k_l\partial_i\omega^{ri}\right)\left(\partial_l\omega_{rk} + \partial_k\omega_{lr} - \partial_r\omega_{kl}\right) =$$
$$= \frac{1}{4}\left(\partial_l\omega^{rk}\partial_l\omega_{rk} + \omega^{kj}\omega^{ri}\partial_j\omega_{li}\left(\partial_l\omega_{rk} + \partial_k\omega_{lr} - \partial_r\omega_{kl}\right)\right) =$$
$$= \frac{1}{4}\left(\partial_l\omega^{rk}\partial_l\omega_{rk} + \omega^{kj}\omega_{rk}\partial_l\omega^{ir}\partial_j\omega_{li} - \partial_r\omega^{jk}\partial_j\omega^{ri}\omega_{li}\omega_{lk}\right) =$$
$$= \frac{1}{4}\left((\omega_{rk}\partial_l\omega^{rk})^2 + \partial_l\omega^{ir}\partial_r\omega_{li} - \partial_r\omega^{jk}\partial_j\omega^{ri}\omega_{li}\omega_{lk}\right). \quad (33)$$

Substituting (31) and (33) into (30) we finally obtain

$$R_{ll} = \frac{1}{4}\left(2\omega^{rk}\partial_l\partial_l\omega_{rk} + (\omega_{rk}\partial_l\omega^{rk})^2 + \partial_l\omega^{ir}\partial_r\omega_{li} - \partial_r\omega^{jk}\partial_j\omega^{ri}\omega_{li}\omega_{lk}\right). \quad (34)$$

Notice that the expression (34) holds for any Fedosov's manifold with skew-symmetric connection.

Then one can find explicit expression for the first diagonal element of the Ricci tensor R_{11} for 2-form given by (6). We omit simple but tedious calculations; the final result is

$$2\omega^{rk}\partial_1\partial_1\omega_{rk} = -4g\cdot\frac{\partial_{q_1}\partial_{q_1}f\cdot(q^ip_i)}{1 - fgq^ip_i} -$$
$$- 8g\cdot\frac{\partial_{q_1}f\left(\partial_{q_1}f\cdot g(q^ip_i)^2 + fg(q^ip_i)p_1 + p_1\right) + f^2gp_1^2}{(1 - fg(q^ip_i))^2}\ ; \quad (35)$$

$$(\omega_{rk}\partial_1\omega^{rk})^2 = 4g^2 \cdot \frac{\big(\partial_{q_1} f \cdot (q^i p_i) + f p_1\big)^2}{(1 - fg(q^i p_i))^2}\ ; \tag{36}$$

$$\partial_1\omega^{ir}\partial_r\omega_{1i} = \frac{f\big(\partial_{q_1} f \cdot (q^i p_i) - f p_1\big)\big(q_3\partial_{p_2} g - q_2\partial_{p_3} g\big)}{(1 - fg(q^i p_i))^2}\ ; \tag{37}$$

$$\partial_r\omega^{jk}\partial_j\omega^{ri}\omega_{1i}\omega_{1k} = \frac{2f\big(\partial_{q_1} f + f^2 g p_1\big)\big(p_2 q_2 + p_3 q_3\big)\big(q_3\partial_{p_2} g - q_2\partial_{p_3} g\big)}{(1 - fg(q^i p_i))^2}. \tag{38}$$

Substituting expressions (35)–(38) into (34) one finally obtains

$$R_{11} = -g \cdot \frac{\partial_{q_1}\partial_{q_1} f \cdot (q^i p_i)}{1 - fgq^i p_i} - g \cdot \frac{\partial_{q_1} f\big(\partial_{q_1} f \cdot g(q^i p_i)^2 + 2p_1\big) + f^2 g p_1^2}{(1 - fg(q^i p_i))^2} +$$
$$+ f \cdot \frac{\Big(q_2\partial_{p_3} g - q_3\partial_{p_2} g\Big)\Big(\partial_{q_1} f\big(p_1 q_1 - p_2 q_2 - p_3 q_3\big) - f p_1\big(1 + 2fg(p_2 q_2 + p_3 q_3)\big)\Big)}{4(1 - fg(q^i p_i))^2}\ , \tag{39}$$

where in the case of magnetic monopole f and g are given by Eqs. (28). To make sure that R_{11} is not zero lets consider the case $p_1 = q_1 = p_3 = q_3 = 0$, then (39) reads

$$R_{11} = \frac{3}{q_2^2(1 - q_2^2 p_2^2)}. \tag{40}$$

As it was mentioned before, if $g = 0$ in (6), then (Hamiltonian is the same) one comes to the same equations (5), although 2-form now is given by Eq. (7). In this case the structure of phase space is more simple. There are only eighteen non-zero connections (we consider the case of magnetic monopole):

$$\Gamma_{i,jk} = \frac{\varepsilon_{ijk}(2q_i^2 - q_j^2 - q_k^2) + \delta_{ij}\varepsilon_{rik} q_r q_i}{(q_1^2 + q_2^2 + q_3^2)^{5/2}}\ , \tag{41}$$

where $i, j, k = 1, 2, 3$; $\Gamma_{i,jk} = -\Gamma^{i+3}_{jk}$.

Curvature tensor has 54 non-zero elements:

$$R^{i+3}_{skl} = \Big(15\big[\delta_{il}\delta_{sk}\big]_{i\to s}\varepsilon_{lkr} q_r q_l q_k + 3\delta_{is}\varepsilon_{skl} q_s(-2q_s^2 + 3q_k^2 + 3q_l^2) +$$
$$+ 3\varepsilon_{rlk} q_r\big[\delta_{is}\delta_{sk}(4q_s^2 - q_r^2 - q_l^2)\big]_{l\to k} +$$
$$+ 3\big[\varepsilon_{ilk}(\delta_{sk} + \delta_{sl}) q_s(4q_i^2 - q_l^2 - q_k^2)\big]_{i\to s}\Big)\big(q_1^2 + q_2^2 + q_3^2\big)^{-7/2}, \tag{42}$$

where $i, s, k, l = 1, 2, 3$; and $\big[a_{ijl}\big]_{i\to j} = a_{ijl} + a_{jil}$.

All elements of the Ricci tensor are zero.

5 Conclusion

The Lagrangian mechanics is included in the Hamiltonian one as the special case (with the help of the Legandre transformation); generally the Hamiltonian mechanics radically differs from the Lagrange one. There are twice as much variables in the Hamiltonian mechanics as in the Lagrangian one; Hamiltonian and independent 2-form are defined in Hamiltonian mechanics. It is impossible to obtain equation (1) in the framework of the Lagrangian mechanics [1], although in the proper choice of α if describes motion of charged particle in the field of magnetic monopole (8). In the simplest generalization of Hamiltonian mechanics (by choosing 2-form) one can obtain equations (5), which are equivalent to (1). This generalization is non-symplectic ($\mathrm{d}\omega^2 \neq 0$), i.e. the Jacoby identity is not fulfilled for the Poissons brackets. One can present corresponding 2-form as the sum of standard symplectic form and some form defined in the configuration space (7), that is equivalent to inclusion of the field of magnetic monopole. There is no Darboux transformation in this theory, variational principle is not known, but the theory is of interest because it allows us to include into consideration "non-Lagrange" systems.

The same equations (5) are obtained in the framework of standard Hamiltonian mechanics, owing to modification of phase space $M^6 \to M^{12}$ with introduction of non-physical variables into the theory, i. e. going theories with constrains. At the same time there are two models: in the first one Hamiltonian is zero when taking constrains into consideration, in the second one it is non-zero.

Generalization of the Fedosov manifold into the case of non-closed 2-forms (non-symplectic manifolds) is constructed (see Sec.3). In this case connections cannot be symmetric. In this case phase space has non-zero torsion tensor (19). As in the case of the standard Fedosov manifolds scalar curvature is zero.

Structure of phase space of such system as charged particle in the field of magnetic monopole is studied (see Sec.4). Curvature tensor for the given system is not zero (see Eq. (27)). The structure of phase space depends on the choice between 2-form (7) and 2-form given by matrix (6), although equations of motion (5) do not change. In the first case all elements of the Ricci tensor are zero, in the second one they are nonzero, see (39), (40). Moreover, for 2-form (7) we find all nonzero connections (41) and all nonzero elements of the curvature tensor (42).

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